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Heat Transfer

Calculation of Radiant Heat Exchange

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Calculation of Radiant Heat Exchange by the Monte Carlo Method

by John R. Howell
NASA Lewis Research Center
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FOREWORD

This Monograph was produced in a pilot program at Oklahoma State University in Stillwater, Oklahoma, under contract to the NASA Office of Technology Utilization. The program was organized to determine the feasibility of presenting the results of recent research in NASA Laboratories and under NASA contract in an educational format suitable as supplementary material in classwork at engineering colleges. The Monograph may result from editing single technical reports or synthesizing several technical reports resulting from NASA's research efforts.

Following the preparation of the Monographs, the program includes their evaluation as educational material in a number of universities throughout the country. The results of these individual evaluations in the classroom situation will be used to help determine if this procedure is a satisfactory way of speeding research results into engineering education.

ABSTRACT

The Monte Carlo Method of solving radiant heat transfer problems basically consists of following groups of photons around through a system until they are either absorbed or lost. By using a large number of photon groups the statistical behavior of the large group will approach the behavior of an actual system. This Monograph discusses the technique required to select photon groups, such that a given statistical distribution will be achieved. An example problem is included, which shows how the Monte Carlo technique can be used to solve problems where energy is emitted and reflected in a non-diffuse or non-specular method. In particular it is assumed that the Fresnel type surface is present. The Fresnel surface distribution is used as an example problem.

INSTRUCTOR'S GUIDE FOR MONOGRAPHS

1. Educational level of the Monograph--Senior or beginning graduate level.

2. Prerequisite course material--The course material required for this Monograph consists of Introductory Material in Heat Transfer, Introductory Material in Differential Equations and Radiation Heat Transfer Concepts up to Configuration Factors and Fresnel Equations.

3. Estimated number of lecture periods required--One hour of lecture.

4. Technical significance of the Monograph--The material presented in this Monograph represents the most advanced numerical technique for the solution of radiation exchange problems. Monte Carlo methods have been applied to the extremely difficult radiation problems involving non-diffuse surfaces and real gases. By using the techniques described herein almost any heat transfer problem can be solved at least in principle. The main difficulty involved is that very much computer time may be required in very complicated real problems.

5. New concepts or unusual concepts illustrated--This Monograph presents the Monte Carlo technique as originally developed for the study of nuclear shielding problems, applied to radiation heat transfer.

6. How Monographs can best be used--

   (a) It is suggested that approximately a one hour lecture be given over the Monograph material.

   (b) It is suggested that the class be assigned a home problem as indicated at the end of the Monograph which involves the use of a digital computer.

   (c) If possible, the students should be allowed to compare the results of their Monte Carlo type solution to a problem solved by another technique.

7. Other literature, Briefs or Monographs of interest--Monte Carlo Solution of Thermal Transfer Through Radiant Media Between Gray Wall, by J. R. Howell and M. Perlmutter, ASME Jour. Heat Trans. 86, no. 2, May 1964, pp. 169-179.: Radiant Transfer Through a Gray Gas Between Concentric Cylinders Using Monte Carlo, by M. Perlmutter, and J. R. Howell, ASME Journ. Heat Trans. 86, no. 2, May 1964, pp. 169-179 and: Radiative Interchange Factors by Monte Carlo by M. M. Weiner, et al. ASME Paper 65-WA/HT-51.

8. Other reports reviewed by the editor in preparing this Monograph--

none.

9. Who to contact for further information--Technical Utilization Officer, Lewis Research Center, Cleveland, Ohio.

10. Note to Instructor: All uncolored pages of the instructors Monograph are in the copies intended for student use.

MONTE CARLO METHOD

Monte Carlo techniques were developed as a way of treating problems in which the happenings at a given location are known, at least in the form of statistical distributions, but in which the equations that describe the interaction between locations are extremely difficult to solve. One example is the local neutron flux produced by the diffusion of neutrons in the core, reflector, and shielding of a nuclear reactor. Because the neutrons arising from the fission process undergo different sequences of absorption, fission, and scattering, a solution for the neutron flux at all points in a heterogeneous system can be difficult. However, the frequency of events occurring along the path of an individual neutron are fairly well understood. This leads to the idea of following sample neutrons and determining the events along their paths by picking events at random from the appropriately weighted set of possibilities at each point. By letting each neutron sample represent a group of real neutrons, and by following enough samples, the flux at each point can be determined.

Of course, such a large number of simple calculations must be performed that a digital computer becomes a necessity. However, the fact that Monte Carlo depends on a large number of simple repetitive calculations and decisions means that it is ideally suited for the computer.

Application to Thermal Radiation

Applying the Monte Carlo method to thermal radiative transfer problems involves setting up a physical model which characterizes radiative processes. In neutron diffusion, sample neutrons obeying known scattering, absorption, and fission laws meet this need. In radiation, the corollary sample particle is the photon. However, if

the photon itself is chosen as a particle to sample, the problem arises that the wavelength of the photon depends on its energy. It is more convenient to choose a common energy for all samples to follow Using this technique, each sample then becomes a bundle of photons with the same wavelength. Each sample can have a different number of photons according to its wavelength. However, the total energy carried by every bundle is the same.

With these bundles of energy as our samples, it becomes relatively straight-forward to simulate radiative processes.

For example, examine the energy transfer between element dA_1 at temperature T_1 and surface A_2, an infinite plane, at temperature $T_2 = 0$, Figure 1. So that some previous statements about directional and spectral properties will gain substance let element dA_1 have emittance

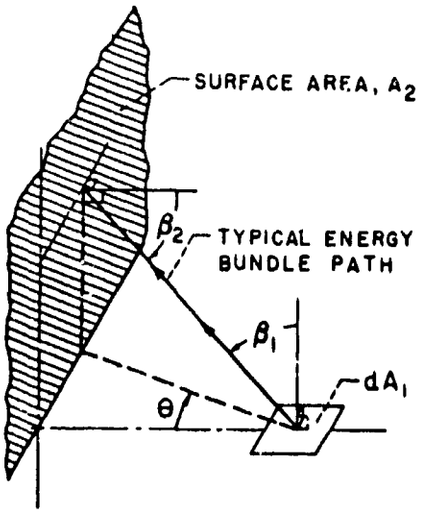


Figure 1. Geometry for example problem

$$\epsilon_1 = \epsilon_1(\lambda, \beta) \tag{1}$$

and let area 2 have emittance

$$\epsilon_2 = \epsilon_2(\lambda, \beta) \tag{2}$$

and assume only that the emittance of both surfaces is independent of circumferential angle θ (Figure 1). This is the case for real surfaces prepared by sandblasting, plating, or etching.

For such a surface, the total emitted energy per unit time is

$$Q_{e,1} = \epsilon_{T,1}\, \sigma T_1^4 dA_1 \tag{3}$$

where $\epsilon_{T,1}$ is the total hemispherical emittance given in this case by

$$\epsilon_{T,1} = \frac{2\pi \int_0^{\infty} \int_0^{\pi/2} \epsilon_1 I_{\lambda,1} \sin\beta \cos\beta \, d\beta \, d\lambda}{\sigma T_1^4} \tag{4}$$

and $I_{\lambda,1}$ is the Planck spectral distribution of black-body radiant intensity.

If it is assumed that $Q_{e,1}$, the total energy emitted per unit time by dA_1, is composed of N of the energy bundles described previously, then the energy of each bundle, c, is simply

$$c = \frac{Q_{e,1}}{N} \tag{5}$$

To determine the energy transferred from element dA_1 to surface A_2, we now follow N bundles of energy through their emission from dA_1, and determine the number S_2 absorbed at A_2. If the energy reflected from A_2 back to dA_1 is neglected, the energy transferred per unit time from dA_1 to A_2 will be

$$Q_{1-2} = cS_2 \tag{6}$$

The next question is how is each individual bundle path determined and how is a wavelength assigned to each bundle? However this is done, the directions and wavelengths of the N bundles must conform to the constraints given by the emittance of the surface and the laws governing radiative processes. For example, if we assign wavelengths to N bundles, the spectral distribution of emitted energy generated by the Monte Carlo process (comprised of the $cN_\lambda \, d\lambda$ for discrete intervals $d\lambda$) must closely approximate the spectrum of the actual emitted energy (plotted as $\pi\epsilon_\lambda I_\lambda d\lambda$ versus λ). To ensure this, a number of methods are available for choosing the energy-bundle properties [1, 2][1]. Leave the radiation problem monentarily for a

1 Numbers in brackets denote references

physical interpretation of two common methods of choosing sample properties.

Choosing Individual Events

Of the two methods to be outlined for randomly selecting events in a manner that obeys the physical constraints of the problem, the first is probably the most intuitively satisfying. It consists of choosing events directly from the curve of known probability of an event.

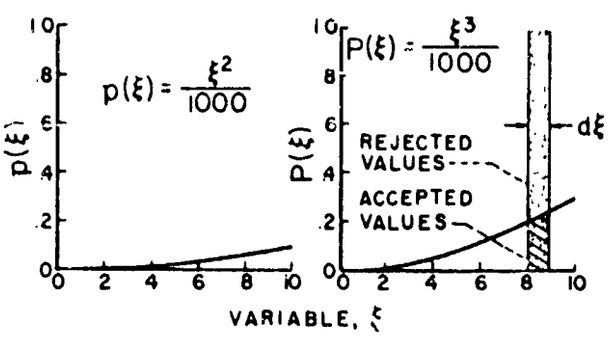
Consider the probability distribution of Figure 2(a), given by the relation

$$p(\xi) = \frac{\xi^2}{1000} \tag{7a}$$

in the interval $0 \leq \xi \leq 10$ and $p(\xi) = 0$ elsewhere. Normalizing this relation by the area under the curve of Figure 2(a) gives

$$p(\xi) = \frac{p(\xi)}{\int_0^{10} p(\xi)d\xi} = \frac{3\xi^2}{1000} \tag{7b}$$

Such a normalized probability curve is called a probability density function.



(a) Probability distribution of variable ξ. (b) Probability density of variable ξ.

Figure 2. Examples of probability distributions

To choose values of ξ in such a manner that Equation (7b) is satisfied by the distribution of chosen values, proceed as follows:

Two numbers, R_a and R_b, are chosen at random from a large set of

numbers evenly distributed in the range (0-1). On t'e digital computer, such R values are selected randomly by means of either a random number generating subroutine or from stored sequences of random numbers. Choosing R values at random ensures that each event in a histroy will be independent of preceding events.

The two random numbers are then used to select a point $P(\xi)$, ξ on Figure 2(b) by setting

$$P(\xi) = R_a ; \quad \xi = R_b(\xi_{max} - \xi_{min}) = 10R_b$$

This point is then compared to the value of $P(\xi)$ at ξ computed from Equation (7b). If the randomly selected value lies above the computed value of $P(\xi)$, then the randomly selected value of ξ is rejected and two new random numbers are selected. Otherwise, the value ξ that has been found is used. Referring again to Figure 2(b), it is seen that such a procedure ensures that the correct fraction of ξ values selected for use will lie in each increment $d\xi$ after enough selections are made.

The difficulty with such an event choosing procedure is that in some cases a large portion of the values of ξ may be rejected. A more efficient method of choosing ξ is therefore desirable. The method to be outlined is more efficient for many of the distributions which occur in radiative-transfer Monte Carlo calculations.

This method is, in short, to integrate the probability density function $P(\xi)$ using the relation

$$R = \int_{-\infty}^{\xi} P(\xi')d\xi' \tag{8}$$

where R can only take on values in the range (0-1) because of the properties of $P(\xi)$. Equation (8) is known as the cumulative distribution function. The function R is then taken to be a random number, and values of ξ are obtained by choosing R at random and solving Equation (8) for the corresponding value of ξ. To show that the probability density of ξ chosen in this way corresponds to the required $P(\xi)$, we can again examine the probability density function of Figure 2(b).

Inserting the example $P(\xi)$ of Equation (7b) into Equation (8) gives

$$R = \int_0^{\xi} P(\xi')d\xi' = \frac{\xi^3}{1000} \tag{9}$$

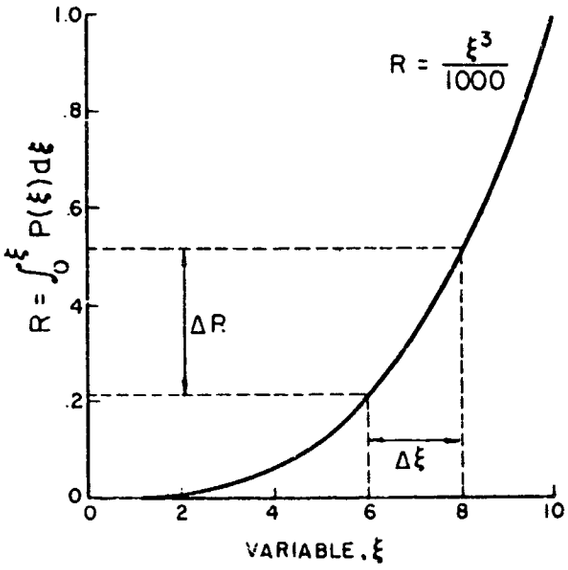


Figure 3. Cumulative distribution of example

Equation (9) is shown plotted in Figure 3. Divide the range of ξ into a number of equal increments $\Delta\xi$. Suppose M values of R are now chosen in the range 0-1, and these M values are picked at equal intervals along R. There will be M values of ξ which correspond to these M values of R. The fraction of the M values of ξ which occurs per given increment $\Delta\xi$ is then

$$\left(\frac{M_{\Delta\xi}}{M}\right)\Delta\xi = \frac{\Delta R}{\Delta\xi} \tag{10}$$

But $\Delta R/\Delta\xi$ is of course an approximation to $(dR/d\xi)$ if a large enough value is chosen for M and small increments $\Delta\xi$ are examined. But $dR/d\xi$ can be seen from Equation (7) to be simply $P(\xi)$ and it has been shown, therefore, that by choosing values of ξ in this manner the required probability distribution is indeed satisfied.

A similar procedure for use when the probability distribution is a nonseparable function of more than one variable is demonstrated in reference [1]; for example, a reflectivity, which for real surfaces may have the probabilities of reflection into the angles of reflection (β,θ) depending on one another.

Selection of Events for the Example Problem

To return to the problem at hand, the wavelength of emission for the emitted bundle must be chosen.

It is assumed here that the surface properties are product functions of the two variables angle and wavelength, that is

$$\epsilon(\lambda,\beta) = \epsilon(\lambda)\ \epsilon(\beta)$$

This assumption is valid for many actual surfaces, since the wavelength variation of emittance, for example, rarely depends on angle of emission. It follows, therefore, that dependence on either variable may be found by integrating out the other variable. Then the normalized probability of emission occuring in the interval $d\lambda$ is

$$P(\lambda) = \int_o^{\pi/2} P(\lambda,\beta)d\beta = \frac{2\pi\int_o^{\pi/2} \epsilon_1(\lambda,\beta)I_\lambda \sin\beta\cos\beta\, d\beta\, d\lambda}{\epsilon_T\sigma T_1^4} \tag{11}$$

Substituting into Equation (8) gives

$$R_\lambda = \frac{2\pi\int_o^{\lambda}\int_o^{\pi/2}\epsilon_1(\lambda',\beta)I_{\lambda'}\sin\beta\cos\beta\, d\beta\, d\lambda'}{\epsilon_T\sigma T_1^4} \tag{12}$$

If the number of bundles N is very large, and this equation was solved for λ each time an R_λ was chosen, computing time could become too large for practical calculations. To circumvent this problem, equations like Equation (12) can be numerically integrated once over the range of λ, and a curve can be fitted to the result. A polynomial curve is often adequate, as is the case in this problem, giving

$$\lambda = A + BR_\lambda + CR_\lambda^2 + \ldots \tag{13}$$

This equation rather than Equation (12) is used in the problem-solving program.

Following a similar procedure for the cone angle of emission β gives the relation

$$R_\beta = \int_o^{\beta}\int_o^{\infty} P(\beta',\lambda)d\lambda\, d\beta' =$$

$$\frac{2\pi\int_o^{\beta}\int_o^{\infty}\epsilon_1(\lambda,\beta')I_\lambda \sin\beta'\cos\beta'\, d\lambda\, d\beta'}{\epsilon_T\sigma T_1^4} \tag{14}$$

which can again be curve-fit to give

$$\beta = A' + B'R_\beta + C'R_\beta^2 + \ldots \tag{15}$$

For gray, diffuse surfaces, Equation (12) reduces to

$$R_{\lambda,gray} = \frac{\pi \int_o^\lambda I_{\lambda'} d\lambda'}{\sigma T_1^4} = F_{o-\lambda} \tag{16}$$

where $F_{o-\lambda}$ is the well-known fraction of black-body emission in the wavelength interval $(0-\lambda)$. Equation (14) for this case reduces to

$$R_{\beta,gray} = 2\int_o^{\beta_1} \sin \beta' \cos \beta' \, d\beta' = \sin^2 \beta_1 \tag{17}$$

or

$$\sin \beta_1 = R_\beta \tag{18}$$

The point to be made here is that computational difficulty in obtaining λ from either Equation (13) or (16) is not greatly different, nor is it much different for obtaining β_1 from either Equation (15) or (18). The difference is mainly in the auxiliary numerical integrations of Equations (12) and (14), which are performed once to get the curve fits for the nongray-nondiffuse solution. As far as the problem solving program is concerned, the more difficult case may just as well be solved. Thus, increasing problem complexity leads to only gradual increases in computer time.

For emission of an individual energy bundle from surface dA_1, then, a wavelength λ can be chosen from Equation (13), and a cone angle of emission β_1 can be chosen from Equation (15). There remains only specification of the circumferential angle θ_1. Because of the assumption made earlier that emission did not depend on θ_1, it is easily shown by the formalism outlined, and is also fairly obvious from intuition, that θ_1 can be determined by

$$\theta_1 = 2\pi R_\theta \tag{19}$$

Because the position of plane A_2 with respect to dA_1 is known, it is a simple matter to determine whether a given energy bundle will strike A_2 after leaving dA_1 in direction (β_1,θ_1). It will hit A_2 if $\cos \theta_1 > 0$, as shown in Figure 1. If it misses, another bundle must

be emitted from dA_1. If the bundle strikes A_2, it must be determined whether it is absorbed or reflected. To do this, geometry is used to find the angle of incidence β_2 of the bundle onto A_2.

$$\cos \beta_2 = \sin \beta_1 \cos \theta_1 \tag{20}$$

Knowing the absorptance of A_2 from Kirchoff's Law

$$\alpha_2(\lambda,\beta) = \epsilon_2(\lambda,\beta) \tag{21}$$

and having determined the wavelength λ of the incident bundle from Equation (13) and the incident angle β_2 from Equation (15), the probability of absorption of the bundle at A_2 can be determined. The probability of absorption is simply the absorptance of A_2 evaluated at β_2 and λ because the directional spectral absorptance $\alpha_2(\lambda,\beta)$ is the fraction of energy incident on A_2 (in a given wavelength interval) from a given solid angle that is absorbed by the surface. The absorptance is therefore the probability-density function for the absorption of incident energy. It is now easy to determine whether a given incident energy bundle is absorbed by using the first of the two event-choosing methods just outlined; that is, by comparing the surface absorptance $\alpha_2(\lambda,\beta)$, which corresponds to $P(\xi)$, the probability of absorption, with a random number R_a. If Equation (22) is satisfied, the bundle of energy is absorbed and a counter S_2 is increased by one.

$$R_\alpha \leq \alpha_2(\lambda,\beta) \tag{22}$$

Otherwise, the bundle is assumed to be reflected and is henceforth neglected. This neglect is reasonable if the absorptance of A_2 is large, or if the directional reflectivity is such that few bundles are reflected back in their original direction. If not, angles of reflection must be chosen from known directional reflectivities and the bundle followed further along its path until absorption or loss from the system. For the purposes of this example, little is to be gained by following the bundle after reflection from surface A_2 because the derivation of the necessary relations is similar to that already presented. The bundles are therfore neglected.

A new bundle is now chosen and its history followed. This

procedure is continued until all N bundles have been emitted from dA_1. The energy absorbed at A_2 is then calculated from Equation (6).

θ values can be constrained to the range $-\pi/2 < \theta < \pi/2$. If N bundles are emitted in this range, then the calculated heat transfer will be

$$Q_{1-2} = \frac{cS_2}{2} \tag{23}$$

The solution of this problem by Monte Carlo is now complete. An astute observer will note that this example could be solved without much difficulty by standard methods. A more astute observer might note further that extension to only slightly more difficult problems would cause serious consequences for the standard treatments; for example, consider introducing a third surface with directional properties into the problem and accounting for all interactions. On the other hand, the author has found few radiation problems that will not yield to a Monte Carlo approach.

Advantages and Applications of Monte Carlo

Several means are available in the literature for solving radiative transport problems. These methods have been called "standard" or "conventional" methods herein, and include the techniques developed by Poljak, Hottel, Oppenheim, and Gebhart [3,5,6,7] as well as formulation in terms of integral equations. Each of these has advantages for certain types of problems, and all will out-shine the Monte Carlo approach in speed and accuracy over some limited range of radiation calculations that is outlined roughly by the complexity of the problem.

The chief usefulness of Monte Carlo to the thermal radiation analyst lies in this fact: Monte Carlo program complexity increases approximately in proportion to problem complexity for radiative interchange problems. This is an important advantage because conventional methods increase approximately with the square of complexity of the problem, due to the matrix form into which they fall. However, because Monte Carlo is somewhat more difficult to apply to the simplest problems, it is most effective in problems where com-

plex geometries and variable properties must be considered. In complex geometries, Monte Carlo has the advantage that simple relations will specify the path of a given energy bundle, whereas most other methods involve explicit or implicit integrations over surface areas. Such integration becomes difficult when a variety of skewed or curved surfaces are present.

Disadvantages of Monte Carlo Technique

Monte Carlo calculations give results which fluctuate around the real answer because the method is a repetitive experiment on a mathematical model used in place of the actual physical situation. The uncertainty can usually be found by applying the standard statistical tests and can be reduced in the same manner as experimental error, that is, by averaging over more tests (bundle histories).

No rigorous test exists to guarantee the convergence of the Monte Carlo results to valid solutions. This has not as yet proven to be a difficulty in thermal radiation problems. It would often be immediately obvious that convergence to invalid solutions was occurring because of the limiting solutions and physical constraints which are known for most radiation problems.

Conclusions

Monte Carlo is discussed in the preceding section as a method suitable for use in the solution of complex problems in radiative transfer. A sample problem is outlined to demonstrate its application and some of the advantages of the technique are discussed along with pertinent literature references.

From all this, certain conclusions emerge. Monte Carlo appears to have a definite advantage over other radiative-transfer calculation techniques when the difficulty of the problem being treated lies above some undefined level. Just where this level is cannot be established, probably being a function of the experience, competence, and prejudice of the individual working the problem. However, problems above this nebulous benchmark in complexity can be treated with greater flexibility, simplicity, and speed. Monte Carlo does lack a kind of generality common to other approaches in that each

problem may require an individual technique, and a dash of ingenuity often helps. This places a greater burden on the programmer's backlog of experience and intuition where standard methods may allow programming through "cookbook" application of their formalism.

For thermal-radiation problems, the parameters and the mathematical relations involved lie in ranges which allow straightforward Monte Carlo programming without the need of the more exotic schemes occasionally necessary in other Monte Carlo transport studies.

With all its advantages, the method suffers from certain problems. The worst of these are the statistical nature of the results, and the lack of guaranteed convergence. It should be noted that the latter fault is common to all methods when complex problems are treated.

Finally, it must be commented that the person using Monte Carlo techniques often develops a physical grasp of the problems encountered because the model being analyzed is simple, and the mathematics describing it are therefore on an unsophisticated basis easily related to the physical model. This is in contrast to the rather poor physical interpretations and predictions which we can make when working with, say, a matrix of non-linear second-order integro-differential equations.

HOME PROBLEM STATEMENT

(a) Determine the heat flux from an elemental area "dA_1" in the horizontal plane to an infinite area "A_2" in the vertical plane, see Figure 1. Consider the surfaces dA_1 and A_2 to be non-conductors and that they are smooth and homogeneous so the Fresnel's equations may be used to predict the properties of the two surfaces. For this problem, the emittance is a function of the polar angle, β, and does not depend on wavelength, λ, or the azimuth angle, θ. The approximation to be used for the emittance and absorptance of both surfaces is as follows:

$$\epsilon = \alpha = A(\cos\beta)^m$$

where

$$A = 0.92$$

and

$$m = 0.24$$

(b) Compare the values obtained from part (a) with the value obtained from the diffuse assumption, $\epsilon \neq \epsilon(\beta)$.

Section

1 - Monte Carlo method

2 - Diffuse assumption

3 - Comparison of the two results

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PROBLEM SOLUTION

Monte Carlo Method

The problem was solved by the procedure described by John Howell. The following assumptions were made:

(1) Emission does not depend on azimuth angle, θ.

(2) Emittance and absorptance are not a function of wavelength, λ.

(3) Emittance and absorptance may be approximated by:

$$\epsilon(\beta) = \alpha(\beta) = 0.92\ (\cos\beta)^{0.24}$$

For this problem, three things must be determined for each bundle of energy leaving dA_1:

(1) Will the emitted bundle of energy strike A_2?

(2) What is the cone angle of emission for each bundle of energy leaving dA_1?

(3) Will the bundle be absorbed by A_2?

(1) As can be seen from Figure 1, the bundle will strike A_2 if $-\pi/2 < \theta_1 < \pi/2$. If the azimuth angle is measured in this manner, the cosine will be positive in the range $-\pi/2 < \theta_1 < \pi/2$. The azimuth angle, θ_1, for each bundle of energy emitted is determined from the following expression:

$$\theta_1 = 2\pi R_\theta$$

where R_θ is a random number in the range of 0 to 1.0. If the cosine of θ_1 is equal to or less than zero, the bundle is rejected since it cannot strike surface A_2.

(2) If the cosine of θ_1 is greater than 0, the cone angle of emission, β_1, is determined from Equation 14 of the Monograph.

$$R_\beta = \int_0^{\beta_1} P(\beta')\, d\beta'$$

where

$$P(\beta_1) = \frac{2\epsilon_1(\beta_1)\ \sin\beta_1\ \cos\beta_1}{\epsilon_H}$$

For this problem

$$\epsilon_1(\beta_1) = 0.92(\cos\beta_1)^{0.24}$$

and

$$\epsilon_H = 2\int_o^{\pi/2} \epsilon_1(\beta')\sin\beta'\cos\beta' d\beta'$$

$$\epsilon_H = 2\int_o^{\pi/2} (0.92)(\cos\beta')^{0.24}\sin\beta'\cos\beta' d\beta'$$

$$\epsilon_H = 2(0.92)\int_o^{\pi/2} \sin\beta'(\cos\beta')^{1.24} d\beta'$$

$$\epsilon_H = 2(0.92)\left[-\frac{(\cos\beta')^{2.24}}{2.24}\right]_o^{\pi/2}$$

$$\epsilon_H = \frac{2(0.92)}{2.24} = 0.8214$$

$$R_\beta = 2\int_o^{\beta_1} P(\beta')d\beta' = 2(0.92)\int_o^{\beta_1} (\cos\beta')^{1.24}\sin\beta' d\beta'$$

$$R_\beta = \frac{2(0.92)\left[-\frac{(\cos\beta')^{2.24}}{2.24}\right]_o^{\beta_1}}{R_\beta} = 1 - (\cos\beta_1)^{2.24}$$

solving for β_1,

$$\cos\beta_1 = (1-R_\beta)^{\frac{1}{2.24}} = (1-R_\beta)^{0.4464}$$

$$\beta_1 = \cos^{-1}(1-R_\beta)^{0.4464}$$

β_1 is the cone angle of emission for the bundle of energy being followed.

(3) To determine if the bundle will be absorbed by surface A_2, the geometry of the system and the absorptance of A_2 is found from

$$\cos\beta_2 = \sin\beta_1\cos\theta_1$$

$$\beta_2 = \cos^{-1}[\sin\beta_1\cos\theta_1]$$

calculating the absorptance of surface A_2,

$$\alpha_2(\beta_2) = 0.92(\cos\beta_2)^{0.24}$$

The above expression, when evaluated at the given β_2, will be the probability of absorption of the energy bundle by surface A_2. In order to determine if the bundle is absorbed, the probability of absorption, $\alpha_2(\beta_2)$, is compared to a random number Ra. If $Ra \leq \alpha_2(\beta_2)$, the bundle of energy is absorbed by surface A_2.

Following is a flow chart of a computer program to perfor the above calculations. The program is written to determine the fraction of the energy emitted by surface dA_1 that will be absorbed by surface A_2.

Computations for Diffuse Assumption

$$\epsilon \neq \epsilon(\beta)$$

From the geometry of the system

$$F_{dA_1 \rightarrow A_2} = 0.5$$

using

$$\alpha_2 = \epsilon_H = 2(0.92) \int_0^{\pi/2} \epsilon(\beta) \sin\beta\cos\beta d\beta$$

$$\alpha_2 = \frac{2(0.92)}{2.24} = 0.8214$$

$\therefore$ the fraction of the energy emitted by dA_1 that will be absorbed by A_2 is

$$\text{Fraction absorbed} = F_{dA_1 \rightarrow A_2} \alpha_2$$

$$= 0.5(0.8214)$$

$$= \underline{0.4107}$$

or 41.07% of the energy emitted by dA_1 will be absorbed by surface A_2.

Comparison of Results

Monte Carlo Method:

energy emitted by dA_1 that will be absorbed = 35.7% by surface A_2.

Diffuse Assumption:

energy emitted by dA_1 that will be absorbed = 41.07%

by surface A_2.

A second counter was placed in the program to determine the fraction of the energy emitted by dA_1 that will strike surface A_2. This was done out of curiosity and as a check on the random number selection by the subroutine. For this particular problem, it is known that 1/2 of the energy leaving dA_1 will strike surface A_2. The computer arrived at a value of 50.48% which is off only 0.48%. This degree of accuracy tends to build confidence in the ability of the subroutine to generate truly random numbers.

The program was run initially using 10,000 bundles of energy leaving dA_1. Since only 1/2 or 5000 bundles were tested for absorption, it was felt that this number was insufficient. The program was run a second time using 20,000 bundles and the results were within 1% of the first run. Through further research on the Monte Carlo technique it was found that the possible error for this method varies inversely with the square root of the number of bundles. Merely increasing the bundles from 10,000 to 20,000 would have very little effect on the results.

The over-all results obtained for the solution to this problem seem very reasonable.

MONTE CARLO FLOW DIAGRAM